

Uniqueness of the coordinate independent $Spin(9) \times SU(2)$ state of Matrix Theory

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Abstract

We explicitly prove, using some nontrivial identities involving gamma matrices, that there can be only one $Spin(9) \times SU(2)$ invariant state which depends only on fermionic variables.

1 Introduction

The explicit construction of the conjectured ground state of the Supermembrane/ M-theory matrix model [1] would certainly bring new ideas into the subject. Although the asymptotic behavior of such state is very well studied [2], not so much is known about the properties of the corresponding wavefunction $\Psi(x)$ near the origin. It is expected that the explicit form of the large x and the small x dependence of $\Psi(x)$ together with some symmetry arguments should fix the entire state uniquely. Following this argument one would like to determine the first few terms of the Taylor expansion of $\Psi(x)$, the problem which was addressed in Ref [3]. Since the wavefunction $\Psi(x)$ must be $Spin(9) \times SU(2)$ invariant [4], the first term of the expansion $\phi := \Psi(0)$

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is the $Spin(9) \times SU(2)$ singlet depending only on the fermionic variables $\theta_{\alpha A}$, $\{\theta_{\alpha A}, \theta_{\beta B}\} = \delta_{AB} \delta_{\alpha\beta}$, ($\alpha = 1, \dots, 16$, $A = 1, 2, 3$). It turns out that ϕ can be expressed in terms of elements of representations of $SO(9)$, in a simple, closed form [3]. This result agrees with earlier approach [5] where ϕ was constructed using a different method. In both [3] and [5], the uniqueness of ϕ is argued relying on the symbolic computer programme. In this paper we give a paper-pencil prove of the uniqueness of ϕ using some novel intertwining relations and identities involving 16×16 gamma matrices in $9 + 1$ dimensions.

2 The construction of ϕ

For fixed color index A , the sixteen fermions $\theta_{\alpha A}$, give rise to the 256 dimensional Hilbert space H_{256} , correspondingly the total Hilbert space H can be written as $H = H_{256} \otimes H_{256} \otimes H_{256}$. The **256** representation of $SO(9)$ is reducible, **256** = **44** \oplus **84** \oplus **128**, and we find that among the possible tensor products, the relevant ones (i.e. those involving the $SO(9)$ singlet) belong to [3, 6]

$$\mathbf{44} \otimes \mathbf{44} \otimes \mathbf{44}, \quad (1)$$

$$\mathbf{84} \otimes \mathbf{84} \otimes \mathbf{84}, \quad (2)$$

$$\mathbf{44} \otimes \mathbf{84} \otimes \mathbf{84}, \mathbf{84} \otimes \mathbf{84} \otimes \mathbf{44}, \mathbf{84} \otimes \mathbf{44} \otimes \mathbf{84}, \quad (3)$$

$$\mathbf{44} \otimes \mathbf{128} \otimes \mathbf{128}, \mathbf{128} \otimes \mathbf{44} \otimes \mathbf{128}, \mathbf{128} \otimes \mathbf{128} \otimes \mathbf{44} \quad (4)$$

and

$$\mathbf{84} \otimes \mathbf{128} \otimes \mathbf{128}, \mathbf{128} \otimes \mathbf{84} \otimes \mathbf{128}, \mathbf{128} \otimes \mathbf{128} \otimes \mathbf{84}. \quad (5)$$

There are in total fourteen $SO(9)$ singlets; 1, 1, 3, 3 and 6 corresponding to (1), (2), (3), (4) and (5) respectively (note the double multiplicity in (5) coming from the fact that **128** \otimes **128** gives two **84**'s while **84** \otimes **84** contains a singlet). An appropriate $SU(2)$ invariant combination of the 14 states yields the desired $Spin(9) \times SU(2)$ singlet.

As shown in [3], among the elements of five representations in (1), (2) and (3) there exists only one such state, explicitly

$$\phi := ||| \underset{44}{1} \rangle + \frac{13}{36} ||| \underset{844}{1} \rangle, \quad (6)$$

$$||| \underset{44}{1} \rangle := |su\rangle_1 |tu\rangle_2 |st\rangle_3,$$

$$|1\rangle_{844} := |suv\rangle_1 |tuv\rangle_2 |st\rangle_3 + |tuv\rangle_1 |st\rangle_2 |suv\rangle_3 + |st\rangle_1 |suv\rangle_2 |tuv\rangle_3,$$

where $|su\rangle_A$ and $|suv\rangle_A$ are the elements of the **44** and the **84** representations, respectively (the elements of (3) do not contribute to ϕ). The main result of this paper is the proof of the fact that in the whole 14 dimensional space of $SO(9)$ singlets, ϕ is a unique $Spin(9) \times SU(2)$ invariant state.

3 The uniqueness

Apart from the five $SO(9)$ singlets considered in [3]

$$S_1 := |su\rangle_1 |tu\rangle_2 |st\rangle_3, \quad (7)$$

$$S_2 := \epsilon^{stupqrabc} |stu\rangle_1 |pqr\rangle_2 |abc\rangle_3, \quad (8)$$

$$S_3 := |suv\rangle_1 |tuv\rangle_2 |st\rangle_3, \quad S_4 := |tuv\rangle_1 |st\rangle_2 |suv\rangle_3, \\ S_5 := |st\rangle_1 |suv\rangle_2 |tuv\rangle_3 \quad (9)$$

corresponding to (1), (2) and (3) respectively, there are 9 additional ones involving the **128** representation. Our choice is

$$S_6 := |s\alpha\rangle_1 |t\alpha\rangle_2 |st\rangle_3, \quad S_7 := |s\alpha\rangle_1 |st\rangle_2 |t\alpha\rangle_3, \\ S_8 := |st\rangle_1 |s\alpha\rangle_2 |t\alpha\rangle_3, \quad (10)$$

$$S_9 := \gamma_{\alpha\beta}^s |u\alpha\rangle_1 |v\beta\rangle_2 |suv\rangle_3, \quad S_{10} := \gamma_{\alpha\beta}^s |u\alpha\rangle_1 |suv\rangle_2 |v\beta\rangle_3, \\ S_{11} := \gamma_{\alpha\beta}^s |suv\rangle_1 |u\alpha\rangle_2 |v\beta\rangle_3, \quad (11)$$

$$S_{12} := \gamma_{\alpha\beta}^{suv} |t\alpha\rangle_1 |t\beta\rangle_2 |suv\rangle_3, \quad S_{13} := \gamma_{\alpha\beta}^{suv} |t\alpha\rangle_1 |suv\rangle_2 |t\beta\rangle_3, \\ S_{14} := \gamma_{\alpha\beta}^{suv} |suv\rangle_1 |t\alpha\rangle_2 |t\beta\rangle_3, \quad (12)$$

(although the states in (11) and (12) have the same representation content, they are linearly independent since there does not exist an identity such as $\delta^{rt}\gamma_{\alpha\beta}^{suv} \propto \delta^{tu}\delta^{rv}\gamma_{\alpha\beta}^s$).

The $SU(2)$ invariance of the linear combination $\tilde{\phi} := \sum_i a_i S_i$, $a_i \in \mathbb{C}$, implies that $J_A \tilde{\phi} = 0$, $A = 1, 2, 3$ where J_A are the $SU(2)$ generators $J_A = \frac{1}{2}\epsilon_{ABC}\theta_{\alpha B}\theta_{\alpha C}$. Let us take $A = 3$ and denote the matrix representation of J_3 by J_{ij} , i.e. $J_3 S_i = \sum_j J_{ji} S_j$. The $SU(2)$ invariance is now equivalent to the matrix equation

$$\sum_i a_i J_{ji} = 0, \quad (13)$$

hence a uniqueness of ϕ is equivalent to the existence of a unique eigenvector of matrix J corresponding to the zero eigenvalue. The first 5 rows of the matrix J_{ij} can be determined from [3]

$$\begin{aligned} J_3 S_1 &= \frac{13}{4} S_6, & J_3 S_2 &= -\frac{3456}{5} S_9 + \frac{972}{5} S_{12}, & J_3 S_3 &= -9 S_6, \\ J_3 S_4 &= \frac{13i}{\sqrt{2}} S_9 - \frac{23i}{\sqrt{2}} S_{12}, & J_3 S_5 &= -\frac{13i}{\sqrt{2}} S_9 + \frac{23i}{\sqrt{2}} S_{12} \quad . \end{aligned}$$

In deriving the above result it is essential to take advantage of the intertwining relations

$$2\theta_{\alpha A}|st\rangle_A = \gamma_{\alpha\beta}^s|t\beta\rangle_A + \gamma_{\alpha\beta}^t|s\beta\rangle_A, \quad (14)$$

$$\theta_{\alpha A}|stu\rangle_A = \frac{i}{\sqrt{2}} \left(\gamma_{\alpha\beta}^{st}|u\beta\rangle_A + \gamma_{\alpha\beta}^{us}|t\beta\rangle_A + \gamma_{\alpha\beta}^{tu}|s\beta\rangle_A \right), \quad (15)$$

note however that, because of the appearance of the **128** in (10), (11) and (12), the evaluation of J_{ij} for $j > 5$ requires one more relation, namely ¹

$$\theta_{\alpha A}|t\beta\rangle_A = \frac{1}{2}\gamma_{\alpha\beta}^\mu|ut\rangle_A - \frac{i}{36\sqrt{2}}\gamma_{\alpha\beta}^{tsuv}|suv\rangle_A - \frac{i}{6\sqrt{2}}\gamma_{\alpha\beta}^{uv}|uvt\rangle_A. \quad (16)$$

After some algebra (see the next section for the details) we find that

$$\begin{aligned} J_3 S_6 &= 4S_1 + \frac{5}{9}S_3, & J_3 S_7 &= \frac{1}{2}S_8 - \frac{4i}{3\sqrt{2}}S_{11} - \frac{5i}{36\sqrt{2}}S_{14}, \\ J_3 S_8 &= \frac{1}{2}S_7 - \frac{4i}{3\sqrt{2}}S_{10} - \frac{5i}{36\sqrt{2}}S_{13}, & J_3 S_9 &= \frac{1}{162}S_2 - \frac{8i}{3\sqrt{2}}S_4 - \frac{8i}{3\sqrt{2}}S_5, \\ J_3 S_{10} &= \frac{7}{6}S_{11} - \frac{1}{18}S_{14}, & J_3 S_{11} &= \frac{7}{6}S_{10} - \frac{1}{18}S_{13}, \\ J_3 S_{12} &= -\frac{11}{162}S_2 + \frac{8i}{3\sqrt{2}}S_4 + \frac{8i}{3\sqrt{2}}S_5, \\ J_3 S_{13} &= -\frac{126i}{\sqrt{2}}S_8 - 62S_{11}, & J_3 S_{14} &= -\frac{126i}{\sqrt{2}}S_7 - 62S_{10}, \end{aligned}$$

¹Eqn. (16) was also known to J.Hoppe and D. Lundholm.

hence

$$[J]_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{162} & 0 & 0 & -\frac{11}{162} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8i}{3\sqrt{2}} & 0 & 0 & \frac{8i}{3\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8i}{3\sqrt{2}} & 0 & 0 & \frac{8i}{3\sqrt{2}} & 0 & 0 \\ \frac{13}{4} & 0 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -63i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -63i\sqrt{2} & 0 \\ 0 & -\frac{3456}{5} & 0 & \frac{13i}{\sqrt{2}} & -\frac{13i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4i}{3\sqrt{2}} & 0 & 0 & \frac{7}{6} & 0 & 0 & -62 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4i}{3\sqrt{2}} & 0 & 0 & \frac{7}{6} & 0 & 0 & -62 & 0 \\ 0 & \frac{972}{5} & 0 & -\frac{23i}{\sqrt{2}} & \frac{23i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{36\sqrt{2}} & 0 & 0 & -\frac{1}{18} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{36\sqrt{2}} & 0 & 0 & -\frac{1}{18} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix J can be easily diagonalized and we find that its kernel is two dimensional spanned by vectors $S_1 + \frac{13}{36}S_3$ and $S_4 + S_5$. Since the singlet must be invariant with respect to any permutation of the color index A , the only possibility is the cyclically invariant combination given by (6).

3.1 Detailed calculation

Below we present the evaluation of $J_3 = \theta_{\alpha 1} \theta_{\alpha 2}$ acting on S_i , $i > 5$ focusing on most important parts of the calculation.

For $i = 6$ the state $J_3 S_6 = \theta_{\alpha 1} \theta_{\alpha 2} |s\beta\rangle |t\beta\rangle |st\rangle$ consists of 9 terms (c.p. (16)), explicitly

$$\begin{aligned} \frac{1}{4} Tr(\gamma^u \gamma^{u_1}) |us\rangle |u_1 t\rangle |st\rangle &= 4S_1, \quad -\frac{i}{72\sqrt{2}} Tr(\gamma^u \gamma^{tpqr}) |us\rangle |pqr\rangle |st\rangle = 0, \\ -\frac{i}{12\sqrt{2}} Tr(\gamma^u \gamma^{pq}) |us\rangle |pqt\rangle |st\rangle &= 0, \quad -\frac{i}{72\sqrt{2}} Tr(\gamma^{spqr} \gamma^u) |pqr\rangle |ut\rangle |st\rangle = 0, \\ -\frac{1}{2592} Tr(\gamma^{spqr} \gamma^{tp_1 q_1 r_1}) |pqr\rangle |p_1 q_1 r_1\rangle |st\rangle &= \frac{1}{9} S_3, \\ \frac{1}{432} Tr(\gamma^{spqr} \gamma^{p_1 q_1}) |pqr\rangle |p_1 q_1 t\rangle |st\rangle &= 0, \\ -\frac{i}{12\sqrt{2}} Tr(\gamma^{pq} \gamma^u) |pqs\rangle |ut\rangle |st\rangle &= 0, \\ -\frac{1}{432} Tr(\gamma^{pq} \gamma^{tp_1 q_1 r_1}) |pqs\rangle |p_1 q_1 r_1\rangle |st\rangle &= 0, \end{aligned}$$

$$-\frac{1}{72}Tr(\gamma^{pq}\gamma^{p_1q_1})|pqs\rangle|p_1q_1t\rangle|st\rangle = \frac{4}{9}S_3,$$

which were evaluated using the following identities

$$Tr(\gamma^u\gamma^{tpqr}) = 0, \quad Tr(\gamma^u\gamma^{pq}) = 0, \quad Tr(\gamma^{spqr}\gamma^u) = 0, \quad Tr(\gamma^{spqr}\gamma^{p_1q_1}) = 0,$$

$$Tr(\gamma^{spqr}\gamma^{tp_1q_1r_1}) = 16 \sum_{\pi \in S_4} sgn(\pi) \delta^{s\pi(t)} \delta^{p\pi(p_1)} \delta^{q\pi(q_1)} \delta^{r\pi(r_1)},$$

$$Tr(\gamma^{pq}\gamma^u) = 0, \quad Tr(\gamma^{pq}\gamma^{tp_1q_1r_1}) = 0, \quad Tr(\gamma^{pq}\gamma^{p_1q_1}) = -16(\delta^{pp_1}\delta^{qq_1} - \delta^{pq_1}\delta^{qp_1}).$$

Therefore we have

$$J_3S_6 = 4S_1 + \frac{5}{9}S_3. \quad (17)$$

For $i = 7$ the state $J_3S_7 = \theta_{\alpha 1}\theta_{\alpha 2}|s\beta\rangle|st\rangle|t\beta\rangle$ consists of 6 terms. They are

$$\begin{aligned} \frac{1}{4}[\gamma^u\gamma^s]_{\beta\beta'}|us\rangle|t\beta'\rangle|t\beta\rangle &= 0, \quad \frac{1}{4}[\gamma^u\gamma^t]_{\beta\beta'}|us\rangle|s\beta'\rangle|t\beta\rangle = \frac{1}{2}S_8 \\ \frac{i}{72\sqrt{2}}[\gamma^{spqr}\gamma^s]_{\beta\beta'}|pqr\rangle|t\beta'\rangle|t\beta\rangle &= -\frac{i}{12\sqrt{2}}S_{14}, \\ \frac{i}{12\sqrt{2}}[\gamma^{pq}\gamma^s]_{\beta\beta'}|pqs\rangle|t\beta'\rangle|t\beta\rangle &= -\frac{i}{12\sqrt{2}}S_{14}, \\ -\frac{i}{72\sqrt{2}}[\gamma^{spqr}\gamma^t]_{\beta\beta'}|pqr\rangle|s\beta'\rangle|t\beta\rangle &= -\frac{i}{\sqrt{2}}S_{11} + \frac{i}{36\sqrt{2}}S_{14}, \\ -\frac{i}{12\sqrt{2}}[\gamma^{pq}\gamma^t]_{\beta\beta'}|pqs\rangle|s\beta'\rangle|t\beta\rangle &= -\frac{i}{3\sqrt{2}}S_{11} \end{aligned}$$

where we used the identities

$$\gamma^{spqr}\gamma^s = -6\gamma^{pqr}, \quad [\gamma^{spqr}, \gamma^t] = 2(\delta^{tr}\gamma^{spq} - \delta^{tq}\gamma^{spr} + \delta^{tp}\gamma^{sqr} - \delta^{ts}\gamma^{pqr})$$

$$[\gamma^{pq}, \gamma^t] = 2\gamma^p\delta^{qt} - 2\gamma^q\delta^{pt}, \quad \gamma^u\gamma^s = \delta^{us}\mathbf{1} + \gamma^{us}$$

the constraint $\sum_s |ss\rangle = 0$ and the Rarita-Schwinger constraint $\gamma_{\alpha\beta}^s|t\beta\rangle = 0$. Therefore we obtain

$$J_3S_7 = \frac{1}{2}S_8 - \frac{4i}{3\sqrt{2}}S_{11} - \frac{5i}{36\sqrt{2}}S_{14} \quad (18)$$

The evaluation of J_3S_8 is analogous to J_3S_7 and we find that

$$J_3S_8 = \frac{1}{2}S_7 - \frac{4i}{3\sqrt{2}}S_{10} - \frac{5i}{36\sqrt{2}}S_{13} \quad (19)$$

For $i = 9$ the state $J_3 S_9 = \theta_{\alpha'1} \theta_{\alpha'2} \gamma_{\alpha\beta}^s |u\alpha\rangle |v\beta\rangle |suv\rangle$ consists of 9 terms. They are

$$\begin{aligned}
\frac{1}{4} \text{Tr}(\gamma^p \gamma^q \gamma^r) |ps\rangle |rt\rangle |qst\rangle &= 0, \quad -\frac{i}{72\sqrt{2}} \text{Tr}(\gamma^{pqrs} \gamma^t \gamma^u) |qrs\rangle |uv\rangle |tpv\rangle = 0, \\
-\frac{i}{12\sqrt{2}} \text{Tr}(\gamma^{pq} \gamma^r \gamma^s) |pqt\rangle |su\rangle |rtu\rangle &= -\frac{8i}{3\sqrt{2}} S_4, \\
-\frac{i}{72\sqrt{2}} \text{Tr}(\gamma^p \gamma^q \gamma^{rstu}) |pv\rangle |stu\rangle |qvr\rangle &= 0, \\
\frac{1}{2592} \text{Tr}(\gamma^{pqrs} \gamma^t \gamma^{p_1 q_1 r_1 s_1}) |qrs\rangle |q_1 r_1 s_1\rangle |tpp_1\rangle &= \frac{1}{162} S_2, \\
\frac{1}{432} \text{Tr}(\gamma^{pp_1} \gamma^q \gamma^{rstu}) |pp_1 v\rangle |stu\rangle |qvr\rangle &= 0, \\
-\frac{i}{12\sqrt{2}} \text{Tr}(\gamma^r \gamma^s \gamma^{pq}) |ru\rangle |pqt\rangle |uts\rangle &= -\frac{8i}{3\sqrt{2}} S_5, \\
\frac{1}{432} \text{Tr}(\gamma^{pqrs} \gamma^t \gamma^{p_1 q_1}) |qrs\rangle |p_1 q_1 v\rangle |tpv\rangle, \\
\frac{1}{72} \text{Tr}(\gamma^{pq} \gamma^r \gamma^{st}) |pqu\rangle |stv\rangle |ruv\rangle &= 0,
\end{aligned}$$

where we used the identities

$$\begin{aligned}
\text{Tr}(\gamma^p \gamma^q \gamma^r) &= 0, \quad \text{Tr}(\gamma^{pqrs} \gamma^t \gamma^u) = 0, \\
\text{Tr}(\gamma^{pq} \gamma^r \gamma^s) &= \text{Tr}(\gamma^p \gamma^q \gamma^{rs}) = -16(\delta^{rp} \delta^{sq} - \delta^{rq} \delta^{sp}), \\
\text{Tr}(\gamma^p \gamma^q \gamma^{rstu}) &= 0, \quad \text{Tr}(\gamma^{pqrs} \gamma^t \gamma^{p_1 q_1 r_1 s_1}) = 16\epsilon^{pqrstp_1 q_1 r_1 s_1}, \\
\text{Tr}(\gamma^{pp_1} \gamma^q \gamma^{rstu}) &= 0, \quad \text{Tr}(\gamma^{pqrs} \gamma^t \gamma^{p_1 q_1}) = 0, \quad \text{Tr}(\gamma^{pq} \gamma^r \gamma^{st}) = 0.
\end{aligned}$$

Therefore we obtain

$$J_3 S_9 = \frac{1}{162} S_2 - \frac{8i}{3\sqrt{2}} S_4 - \frac{8i}{3\sqrt{2}} S_5. \quad (20)$$

For $i = 10$ the state $J_3 S_{10} = \theta_{\alpha'1} \theta_{\alpha'2} \gamma_{\alpha\beta}^s |u\alpha\rangle |suv\rangle |v\beta\rangle$ consists of 9 terms. We use the Rarita-Schwinger constraint and

$$\gamma^{ts} \gamma^s = 8\gamma^t, \quad \gamma^u \gamma^s = 2\delta^{us} \mathbf{1} + \gamma^{us}, \quad \gamma^{tabc} = \gamma^t \gamma^{abc}, t \neq a, b, c$$

to find that they are

$$\frac{i}{2\sqrt{2}} [\gamma^{pq} \gamma^r \gamma^s]_{\alpha\beta} |rq\rangle |t\alpha\rangle |t\beta\rangle = 0, \quad \frac{1}{72} [\gamma^{pq} \gamma^{qstu} \gamma^p]_{\alpha\beta} |stu\rangle |v\alpha\rangle |v\beta\rangle = -\frac{1}{9} S_{14},$$

$$\begin{aligned}
\frac{1}{12}[\gamma^{pq}\gamma^{rs}\gamma^p]_{\alpha\beta}|rsq\rangle|v\alpha\rangle|v\beta\rangle &= -\frac{1}{3}S_{14}, \quad \frac{i}{2\sqrt{2}}[\gamma^{pq}\gamma^r\gamma^q]_{\alpha\beta}|rs\rangle|s\alpha\rangle|p\beta\rangle = 0, \\
\frac{1}{72}[\gamma^{pq}\gamma^{rstu}\gamma^q]_{\alpha\beta}|stu\rangle|r\alpha\rangle|p\beta\rangle &= 0, \quad \frac{1}{12}[\gamma^{pq}\gamma^{rs}\gamma^q]_{\alpha\beta}|rst\rangle|t\alpha\rangle|p\beta\rangle = 2S_{11}, \\
\frac{i}{2\sqrt{2}}[\gamma^{pq}\gamma^r\gamma^s]_{\alpha\beta}|rp\rangle|s\alpha\rangle|q\beta\rangle &= 0, \quad \frac{1}{72}[\gamma^{pq}\gamma^{prst}\gamma^u]_{\alpha\beta}|rst\rangle|u\alpha\rangle|q\beta\rangle = \frac{2}{9}S_{14}, \\
\frac{1}{12}[\gamma^{pq}\gamma^{rs}\gamma^t]_{\alpha\beta}|rsp\rangle|t\alpha\rangle|q\beta\rangle &= \frac{1}{6}S_{14} - \frac{5}{6}S_{11}.
\end{aligned}$$

Therefore we obtain

$$J_3S_{10} = \frac{7}{6}S_{11} - \frac{1}{18}S_{14}. \quad (21)$$

Evaluation of J_3S_{11} is analogous to J_3S_{10} and we find that

$$J_3S_{11} = \frac{7}{6}S_{10} - \frac{1}{18}S_{13}. \quad (22)$$

For $i = 12$ the state $J_3S_{12} = \theta_{\alpha'1}\theta_{\alpha'2}\gamma_{\alpha\beta}^{suuv}|t\alpha\rangle|t\beta\rangle|suuv\rangle$ consists of 9 terms. They are

$$\begin{aligned}
&\frac{1}{4}Tr(\gamma^p\gamma^{qrs}\gamma^t)|pa\rangle|ta\rangle|qrs\rangle = 0, \\
&-\frac{i}{72\sqrt{2}}Tr(\gamma^{pqrs}\gamma^{p_1q_1r_1}\gamma^u)|qrs\rangle|up\rangle|p_1q_1r_1\rangle = -\frac{16i}{3\sqrt{2}}S_4, \\
&-\frac{i}{12\sqrt{2}}Tr(\gamma^{pq}\gamma^{qst}\gamma^u)|pqt\rangle|tu\rangle|qst\rangle = \frac{8i}{\sqrt{2}}S_4, \\
&-\frac{i}{72\sqrt{2}}Tr(\gamma^p\gamma^{qrs}\gamma^{tq_1r_1s_1})|pt\rangle|q_1r_1s_1\rangle|qrs\rangle = -\frac{16i}{3\sqrt{2}}S_5, \\
&-\frac{1}{2592}Tr(\gamma^{pqrs}\gamma^{tuv}\gamma^{pq_1r_1s_1})|qrs\rangle|q_1r_1s_1\rangle|tuv\rangle = \frac{1}{162}S_2, \\
&-\frac{1}{432}Tr(\gamma^{pq}\gamma^{rst}\gamma^{ur_1s_1t_1})|pqu\rangle|r_1s_1t_1\rangle|rst\rangle = -\frac{1}{27}S_2, \\
&-\frac{i}{12\sqrt{2}}Tr(\gamma^p\gamma^{qrs}\gamma^{tu})|pv\rangle|tuv\rangle|qrs\rangle = \frac{8i}{\sqrt{2}}S_5, \\
&-\frac{1}{432}Tr(\gamma^{pqrs}\gamma^{q_1r_1s_1}\gamma^{tu})|qrs\rangle|tup\rangle|q_1r_1s_1\rangle = -\frac{1}{27}S_2, \\
&-\frac{1}{72}Tr(\gamma^{pq}\gamma^{rst}\gamma^{uv})|pqw\rangle|uvw\rangle|rst\rangle = 0,
\end{aligned}$$

where we used

$$\begin{aligned}
Tr(\gamma^p \gamma^{qr s} \gamma^t) &= 0, \\
Tr(\gamma^{pqrs} \gamma^{p_1 q_1 r_1} \gamma^u) &= Tr(\gamma^u \gamma^{p_1 q_1 r_1} \gamma^{pqrs}) = 16 \sum_{\pi \in S_4} sgn(\pi) \delta^{s\pi(u)u} \delta^{p\pi(p_1)} \delta^{q\pi(q_1)} \delta^{r\pi(r_1)}, \\
Tr(\gamma^{pqrs} \gamma^{tuv} \gamma^{pq_1 r_1 s_1}) &= -48 \epsilon^{qrstuv q_1 r_1 s_1}, \quad Tr(\gamma^{pq} \gamma^{rst} \gamma^{ur_1 s_1 t_1}) = 16 \epsilon^{pqur_1 s_1 t_1 rst}, \\
Tr(\gamma^{rst} \gamma^u \gamma^{pq}) &= Tr(\gamma^{rst} \gamma^{pq} \gamma^u) = -16 \sum_{\pi \in S_3} \delta^{p\pi(r)} \delta^{q\pi(s)} \delta^{u\pi(t)}, \\
Tr(\gamma^{pqrs} \gamma^{q_1 r_1 s_1} \gamma^{tu}) &= 16 \epsilon^{qrstup q_1 r_1 s_1}, \quad Tr(\gamma^{pq} \gamma^{rst} \gamma^{uv}) = 0.
\end{aligned}$$

Therefore we obtain

$$J_3 S_{12} = -\frac{11}{162} S_2 + \frac{8i}{3\sqrt{2}} S_4 + \frac{8i}{3\sqrt{2}} S_5. \quad (23)$$

For $i = 13$ state $J_3 S_9 = \theta_{\alpha'1} \theta_{\alpha'2} \gamma_{\alpha\beta}^{suv} |t\alpha\rangle |suv\rangle |t\beta\rangle$ consists of 3 terms. They are

$$\begin{aligned}
\frac{3i}{2\sqrt{2}} [\gamma^{pq} \gamma^r \gamma^{pqu}]_{\alpha\beta} |rt\rangle |u\alpha\rangle |t\beta\rangle &= -\frac{126i}{\sqrt{2}} S_8, \\
\frac{3}{72} [\gamma^{pq} \gamma^{rstu} \gamma^{pqv}]_{\alpha\beta} |stu\rangle |v\alpha\rangle |r\beta\rangle &= 0, \\
\frac{3}{12} [\gamma^{pq} \gamma^{rs} \gamma^{pqv}]_{\alpha\beta} |rst\rangle |v\alpha\rangle |t\beta\rangle &= -62 S_{11},
\end{aligned}$$

where we used the R-S constraint and

$$\begin{aligned}
[\gamma^{su}, \gamma^v] &= 2\gamma^s \delta^{vt} - 2\gamma^u \delta^{ut}, \quad \gamma^{su} \gamma^{suv} = -56 \gamma^v, \\
\gamma^{tabc} &= \gamma^t \gamma^{abc}, t \neq a, b, c, \quad \gamma^{abct} = \gamma^{abc} \gamma^t, t \neq a, b, c, \quad [\gamma^t, \gamma^{suv}] = 2\gamma^{tsuv}, \\
[\gamma^{su}, \gamma^{ab}] &= 2(\gamma^{sb} \delta^{ua} - \gamma^{sa} \delta^{ub} + \gamma^{ua} \delta^{sb} - \gamma^{ub} \delta^{sa}).
\end{aligned}$$

Therefore we obtain

$$J_3 S_{13} = -\frac{126i}{\sqrt{2}} S_8 - 62 S_{11}. \quad (24)$$

Evaluation of $J_3 S_{14}$ is analogous to $J_3 S_{14}$

$$J_3 S_{14} = -\frac{126i}{\sqrt{2}} S_7 - 62 S_{10}. \quad (25)$$

4 Outlook

The uniqueness of the $Spin(9) \times SU(2)$ state ϕ , proven in this paper, is a starting point for the unique Fock space representation of the hamiltonian of the full matrix model

$$H = K + V + H_F$$

$$K = -\partial_{As}\partial_{As}, \quad V = \frac{1}{2}(\epsilon_{ABC}x_{Bs}x_{Cs})^2, \quad H_F = if_{CAB}\gamma_{\alpha\beta}^s x_{Cs}\theta_{A\alpha}\theta_{B\beta},$$

in terms of $Spin(9) \times SU(2)$ invariant basis. To be more specific consider the normalized "vacuum" state

$$|v\rangle := \frac{1}{\|\phi\|}|0\rangle_B \otimes \phi,$$

where $\|\phi\|^2 = 14014/9$ (explicitly checked on the computer) and $|0\rangle_B$ is the bosonic Fock vacuum (in the coordinate representation $\langle x|0\rangle \propto \exp(-\frac{1}{2}x_{As}x_{As})$). Such state is also $Spin(9) \times SU(2)$ invariant and gives a possibility to represent H in the $Spin(9) \times SU(2)$ invariant basis obtained by acting with bosonic creation operators a_{As}^\dagger and fermionic operators $\theta_{A\alpha}$ on $|v\rangle$.

First step towards this direction can be done by finding the expectation value $\langle v|H|v\rangle$. There is no contribution from H_F since H_F is linear in x_{As} while the contributions from K and V are $27/2$ and 54 respectively, implying that

$$\langle v|H|v\rangle = 67.5,$$

a rather large number considering the existence of the (conjectured) zero-energy ground state of the model.

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